



Existence of Solutions to Singular Integral Equations

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Abstract—Nonnegative solutions are established for singular integral equations of the form $y(t) = h(t) + \int_0^T k(t, s) f(s, y(s)) ds$ for $t \in [0, T]$. Here f may be singular at $y = 0$. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

This paper discusses the singular integral equation

$$y(t) = h(t) + \int_0^T k(t, s) f(s, y(s)) ds, \quad \text{for } t \in [0, T], \quad T > 0 \text{ fixed.} \quad (1.1)$$

Here our nonlinearity $f(t, y)$ may be singular at $y = 0$. Problems of this type, when (1.1) models second- and higher-order boundary value problems, have been discussed extensively in the literature [1–4]. Unfortunately, the more general (i.e., for general kernel k) singular integral equation (1.1) has received only a brief mention. In this paper, we initiate the study of singular integral equations of type (1.1). In addition, we present new and very general existence results for (1.1). For example we will show that

$$y(t) = \int_0^T t^\gamma \sqrt{s+t} \left([y(s)]^{-\alpha} + [y(s)]^\beta + 1 \right) ds, \quad \text{for } t \in [0, T], \quad (1.2)$$

with $\alpha > 0$, $0 \leq \beta < 1$, $\gamma \geq 0$, and $\gamma\alpha < 1$, has a solution $y \in C[0, T]$ with $y > 0$ on $(0, T]$ (notice $y(0) = 0$).

The theory in Section 2 makes use of the following well-known existence principle from the literature [5–7].

THEOREM 1.1. *Let $1 \leq p \leq \infty$ be a constant and q be such that $1/p + 1/q = 1$. Assume*

$$h \in C[0, T], \quad (1.3)$$

$g : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$ is a L^q -Carathéodory function. By this we mean:

- (i). the map $y \mapsto g(t, y)$ is continuous for almost all t in $[0, T]$,
- (ii). the map $t \mapsto g(t, y)$ is measurable for all, y in \mathbf{R} ,
- (iii). for any $r > 0$ there exists $\mu_r \in L^q[0, T]$ such that
 $|y| \leq r$ implies $|g(t, y)| \leq \mu_r(t)$ for almost all t in $[0, T]$,

$$k_t(s) = k(t, s) \in L^p[0, T], \quad \text{for each } t \in [0, T] \quad (1.5)$$

and

$$\text{the map } t \mapsto k_t \text{ is continuous from } [0, T] \text{ to } L^p[0, T] \quad (1.6)$$

hold. In addition, suppose there is a constant $M > |h|_0 = \sup_{[0, T]} |h(t)|$, independent of λ , with $|y|_0 = \sup_{[0, T]} |y(t)| \neq M$ for any solution $y \in C[0, T]$ to

$$y(t) = h(t) + \lambda \int_0^T k(t, s)g(s, y(s)) ds, \quad t \in [0, T], \quad (1.7)_\lambda$$

for each $\lambda \in (0, 1)$. Then

$$y(t) = h(t) + \int_0^T k(t, s)g(s, y(s)) ds, \quad t \in [0, T], \quad (1.8)$$

has at least one solution in $C[0, T]$.

2. SINGULAR PROBLEMS

In this section, we discuss (1.1). The following conditions will be assumed throughout:

$$h \in C[0, T] \text{ with } h \geq 0 \text{ on } [0, T]; \quad (2.1)$$

$$f : [0, T] \times (0, \infty) \rightarrow (0, \infty) \text{ is continuous}; \quad (2.2)$$

$$k(t, s) \geq 0 \text{ a.e. on } [0, T] \times [0, T]; \quad (2.3)$$

$$k_t(s) \in L^p[0, T], \text{ for each } t \in [0, T]; \text{ here } 1 \leq p \leq \infty \text{ is a constant}; \quad (2.4)$$

$$\text{the map } t \mapsto k_t \text{ is continuous from } [0, T] \text{ to } L^p[0, T]; \quad (2.5)$$

$$\begin{aligned} f(t, u) &\leq g(u) + r(u) \text{ on } [0, T] \times (0, \infty) \text{ with } g > 0 \text{ continuous} \\ &\text{and nonincreasing on } (0, \infty) \text{ and } r \geq 0 \text{ continuous and} \\ &\text{nondecreasing on } [0, \infty); \end{aligned} \quad (2.6)$$

$$\begin{aligned} &\text{there exists a function } \psi \geq 0 \text{ continuous on } [0, T] \\ &\text{with } f(t, u) \geq \psi(t) \text{ on } [0, T] \times (0, \infty); \end{aligned} \quad (2.7)$$

there exists a subset I of $[0, T]$ of measure zero with

$$\mu(t) = h(t) + \int_0^T k(t, s)\psi(s) ds > 0, \quad \text{for } t \in [0, T] \setminus I; \quad (2.8)$$

$$\int_0^T g^q(\mu(s)) ds < \infty; \text{ here } \frac{1}{p} + \frac{1}{q} = 1; \quad (2.9)$$

and

$$\text{if } z > 0 \text{ satisfies } z \leq a_0 + b_0 r(z) \text{ for constants } a_0 \geq 0 \text{ and } b_0 \geq 0, \text{ then there} \\ \text{exists a constant } K \text{ (which may depend only on } a_0 \text{ and } b_0) \text{ with } z \leq K. \quad (2.10)$$

THEOREM 2.1. Suppose (2.1)–(2.10) hold. Then (1.1) has a solution $y \in C[0, T]$ with $y > 0$ on $[0, T] \setminus I$.

PROOF. Let $n_0 \in \{1, 2, \dots\} = N_0$. We first show

$$y(t) = \frac{1}{m} + h(t) + \int_0^T k(t, s)f^*(s, y(s)) ds, \quad t \in [0, T], \quad (2.11)^m$$

has a solution for each $m \in N_0$; here

$$f^*(t, u) = \begin{cases} f(t, u), & u \geq \frac{1}{m}, \\ f\left(t, \frac{1}{m}\right), & u \leq \frac{1}{m}. \end{cases}$$

To show (2.11)^m has a solution for each $m \in N_0$ we will apply Theorem 1.1. Consider the family of problems

$$y(t) = \frac{1}{m} + h(t) + \lambda \int_0^T k(t, s)f^*(s, y(s)) ds, \quad t \in [0, T], \quad (2.12)_\lambda^m$$

for $0 < \lambda < 1$. Let $y \in C[0, T]$ be any solution of (2.12)_λ^m. Now (2.1), (2.2), and (2.12)_λ^m imply $y(t) \geq 1/m$ for $t \in [0, T]$.

REMARK 2.1. Similarly any solution $u \in C[0, T]$ of (2.12)₁^m satisfies $u(t) \geq 1/m$ for $t \in [0, T]$.

Also notice that

$$\begin{aligned} |y(t)| &\leq 1 + h(t) + \int_0^T k(t, s)[g(y(s)) + r(y(s))] ds \\ &\leq 1 + |h|_0 + \left[g\left(\frac{1}{m}\right) + r(|y|_0) \right] \int_0^T k(t, s) ds, \end{aligned}$$

for $t \in [0, T]$, and so

$$|y|_0 \leq \left[1 + |h|_0 + g\left(\frac{1}{m}\right) \sup_{t \in [0, T]} \int_0^T k(t, s) ds \right] + r(|y|_0) \sup_{t \in [0, T]} \int_0^T k(t, s) ds.$$

This together with (2.10) implies that there is a constant $K(m)$ with $|y|_0 \leq K(m)$. Consequently, Theorem 1.1 guarantees that (2.11)^m has a solution $y_m \in C[0, T]$ with (see Remark 2.1) $y_m(t) \geq 1/m$ for $t \in [0, T]$, and of course y_m is a solution of

$$y(t) = \frac{1}{m} + h(t) + \int_0^T k(t, s)f(s, y(s)) ds, \quad \text{for } t \in [0, T]. \quad (2.13)$$

Notice as well that (2.7) together with (2.13) yields

$$y_m(t) \geq h(t) + \int_0^T k(t, s) \psi(s) ds = \mu(t), \quad \text{for } t \in [0, T]. \quad (2.14)$$

We shall now obtain a solution to (1.1) by means of the Arzela-Ascoli Theorem, as a limit of solutions of (2.11)^m. To this end we will show

$$\{y_m\}_{m \in N_0} \text{ is a bounded, equicontinuous family on } [0, T]. \quad (2.15)$$

Notice (2.6), (2.13), and (2.14) yield

$$\begin{aligned} |y_m(t)| &\leq 1 + |h|_0 + \int_0^T k(t, s) [g(\mu(s)) + r(y_m(s))] ds \\ &\leq \left[1 + |h|_0 + \int_0^T k(t, s) g(\mu(s)) ds \right] + r(|y_m|_0) \int_0^T k(t, s) ds, \end{aligned}$$

for $t \in [0, T]$, and so

$$|y_m|_0 \leq \left[1 + |h|_0 + \sup_{t \in [0, T]} \int_0^T k(t, s) g(\mu(s)) ds \right] + r(|y_m|_0) \sup_{t \in [0, T]} \int_0^T k(t, s) ds.$$

REMARK 2.2. Notice

$$\sup_{t \in [0, T]} \int_0^T k(t, s) g(\mu(s)) ds \leq \sup_{t \in [0, T]} \left(\int_0^T k^p(t, s) ds \right)^{1/p} \left(\int_0^T g^q(\mu(s)) ds \right)^{1/q}.$$

Now (2.10) implies that there is a constant M (independent of m) with $|y_m|_0 \leq M$. Thus, $\{y_m\}_{m \in N_0}$ is a bounded family on $[0, T]$. To see the remainder of (2.15) notice for $t, x \in [0, T]$ that

$$\begin{aligned} |y_m(t) - y_m(x)| &\leq |h(t) - h(x)| + \int_0^T |k(t, s) - k(x, s)| [g(\mu(s)) + r(M)] ds \\ &\leq |h(t) - h(x)| + \left(\int_0^T |k_t(s) - k_x(s)|^p ds \right)^{1/p} \left(\int_0^T g^q(\mu(s)) ds \right)^{1/q} \\ &\quad + r(M) \int_0^T |k_t(s) - k_x(s)| ds. \end{aligned}$$

The Arzela-Ascoli Theorem guarantees the existence of a subsequence N of N_0 and a function $y \in C[0, T]$ with y_m converging uniformly on $[0, T]$ to y as $m \rightarrow \infty$ through N . Also $y(t) \geq \mu(t)$ for $t \in [0, T]$ so $y(t) > 0$ for $t \in [0, T] \setminus I$. In addition,

$$y_m(t) = \frac{1}{m} + h(t) + \int_0^T k(t, s) f(s, y_m(s)) ds, \quad \text{for } t \in [0, T]. \quad (2.16)$$

Now $f : [0, T] \times (0, \infty) \rightarrow (0, \infty)$ continuous, together with $\mu \in C[0, T]$, $\mu(t) > 0$ for $t \in [0, T] \setminus I$ and $y(t) \geq \mu(t)$ on $[0, T]$ implies

$$f(s, y_m(s)) \rightarrow f(s, y(s)), \quad \text{for each } s \in [0, T] \setminus I.$$

Also notice for fixed a.e. $t \in [0, T]$,

$$|k(t, s) f(s, y_m(s))| \leq k(t, s) \{g(\mu(s)) + r(M)\} \in L^1[0, T].$$

Fix $t \in [0, T]$. Let $m \rightarrow \infty$ through N in (2.16) to obtain (here we use the Lebesgue dominated convergence theorem),

$$y(t) = h(t) + \int_0^T k(t, s) f(s, y(s)) ds. \quad \blacksquare$$

EXAMPLE. Consider the integral equation

$$y(t) = \int_0^T k(t, s) \left([y(s)]^{-\alpha} + \eta [y(s)]^\beta + 1 \right) ds, \quad t \in [0, T] \quad (2.17)$$

with $\alpha > 0$, $0 \leq \beta < 1$, and $\eta \geq 0$. Suppose (2.3)–(2.5) hold and in addition assume

$$\begin{aligned} &\text{there exists a subset } I \text{ of } [0, T] \text{ of measure zero with} \\ &\mu(t) = \int_0^T k(t, s) ds > 0, \quad \text{for } t \in [0, T] \setminus I \end{aligned} \quad (2.18)$$

and

$$\int_0^T [\mu(s)]^{-\alpha q} ds < \infty, \quad \text{here } \frac{1}{p} + \frac{1}{q} = 1 \quad (2.19)$$

are satisfied. Then (2.17) has a solution $y \in C[0, T]$ with $y > 0$ on $[0, T] \setminus I$.

The result follows immediately from Theorem 2.1 since (2.1), (2.2), and (2.6) (with $g(u) = u^{-\alpha}$ and $r(u) = \eta u^\beta + 1$), (2.7) (with $\psi = 1$), and (2.10) (since $0 \leq \beta < 1$) hold.

REMARK 2.3. Notice (1.2) has a solution $y \in C[0, T]$ with $y(t) > 0$ for $t \in (0, T]$ since $\gamma\alpha < 1$. This follows from the above example with $\eta = 1$ and $k(t, s) = t^\gamma \sqrt{s+t}$. Notice $I = \{0\}$, $p = 1$, $q = \infty$, and $\mu(t) = t^\gamma \int_0^t \sqrt{s+t} ds$.

REFERENCES

1. R.P. Agarwal and D. O'Regan, Nonlinear superlinear singular and nonsingular second order boundary value problems, *Jour. Diff. Eqns.* **143**, 60–95, (1998).
2. R.P. Agarwal and D. O'Regan, Positive solutions for $(p, n - p)$ conjugate boundary value problems, *Jour. Diff. Eqns.* (to appear).
3. P.W. Eloe and J. Henderson, Existence of solutions for some higher order boundary value problems, *ZAMM* **73**, 315–323, (1993).
4. D. O'Regan, *Theory of Singular Boundary Value Problems*, World Scientific Press, Singapore, (1994).
5. D. O'Regan and M. Meehan, *Existence Theory for Nonlinear Integral and Integrodifferential Equations*, Kluwer Acad. Publ. (to appear).
6. D. O'Regan, Existence results for nonlinear integral equations, *J. Math. Anal. Appl.* **192**, 705–726, (1995).
7. D. O'Regan, Existence theory for nonlinear Volterra and Hammerstein integral equations, In *Dynamical Systems and Applications*, (Edited by R.P. Agarwal), pp. 601–615, Vol. 4, World Scientific Series in Applicable Analysis, River Edge, NJ, (1995).
8. C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, New York, (1990).
9. G. Gripenberg, S.O. Londen and O. Staffans, *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, New York, (1990).